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FIELDS NEAR A RAPIDLY PROPAGATING CRACK TIP IN AN ELASTIC PERFE--ETC(U)

JAN 80 J D ACHENBACH, V DUNAYEVSKY

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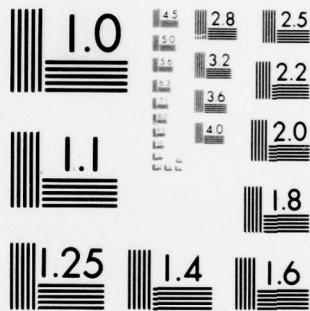
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FIELD'S NEAR A RAPIDLY PROPAGATING CRACK TIP  
IN AN ELASTIC PERFECTLY-PLASTIC MATERIAL

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dynamic solutions describe a region of plastic loading which completely surrounds the propagating crack tip.

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### Introduction

Fast fracture induces rapid particle motions in the cracking body. From the point of view of fracture mechanics such dynamic effects are of particular interest in the immediate vicinity of the propagating crack tip, where the fields of stress and deformation are critical for either continued fracture or for crack arrest.

A substantial body of literature has dealt with dynamic effects on essentially brittle fracture, within the context of linear elastic fracture mechanics. We mention review articles by Achenbach (1972), Freund (1975) and Kanninen (1978). Dynamic effects in the presence of elastic-plastic constitutive behavior have, however, been considered in only a few studies.

An investigation of the dynamic near-tip fields in an elastic perfectly-plastic material was presented by Slepyan (1976), who considered both the cases of anti-plane and in-plane strains. Dynamic near-tip effects in the presence of strain hardening were investigated by Achenbach and Kanninen (1978) on the basis of  $J_2$  flow theory. These authors found results which are very similar to the ones obtained by Amazigo and Hutchinson (1977) for the corresponding quasi-static problem. As shown by Achenbach, Burgers and Dunayevsky (1979), for strain hardening the governing equations are elliptic when the crack-tip speed is less than a certain critical value. The usual separation-of-variables asymptotic analysis can then be carried out, which yields singularities of the general type  $r^p$  ( $-1 < p < 0$ ) for the stresses and the strains. As the crack-tip speed increases (or alternatively as the strain-hardening curve becomes flatter) the nature of the governing equations becomes, however, hyperbolic, and the near-tip fields appear to change

character. Indeed in the limit of elastic perfectly-plastic behavior the stresses become bounded and only some strains display singularities, as shown in the present paper, and earlier by Slepyan (1976).

In this paper dynamic effects on near-tip fields are investigated for elastic perfectly-plastic constitutive behavior. The approach is different from the one employed by Slepyan (1976), but the results are identical for the Mode-III case. For Mode-I the results show some differences. In the present paper the stresses and the strains have been expressed in explicit form, and they include higher order terms.

As the crack-tip speed decreases the expressions for the stresses reduce to the ones for the corresponding quasi-static problem, as might be expected on the basis of intuitive arguments. This is, however, not true for the strains, which become unbounded in the limit of vanishing crack-tip speed. A significant difference between the dynamic and quasi-static solutions is that the dynamic solution does not describe elastic unloading behind the crack tip. These anomalies suggest that the transition from dynamic to quasi-static conditions is non-uniform. It is speculated that the transition from the dynamic to the quasi-static solution with decreasing crack-tip speed is effected because the dynamic solution is asymptotically valid in a small zone, which shrinks on the crack tip in the limit of vanishing crack-tip speed.

### 1. Governing Equations

Both a stationary coordinate system with axes denoted by  $x_i$ , and a moving coordinate system with axes denoted by  $(x,y,z)$  are considered. The moving coordinate system has its origin at the propagating crack tip. The geometry is shown in Fig. 1. In this section the equations governing the

motions of an elastic perfectly-plastic material are stated in the stationary coordinate system. In the next sections these equations are simplified for anti-plane strain and plane strain, for the special case of "steady-state" fields of stress and deformation relative to the moving crack tip.

Relative to the stationary coordinate system the equations of motion are

$$\partial_j \sigma_{ij} = \rho \partial_t^2 u_i \quad (1.1)$$

In the zone of plastic deformation the stresses are assumed to satisfy the Tresca yield condition, which states that

$$|\tau|_{\max} = k \quad (1.2)$$

where  $|\tau|_{\max}$  is the maximum shear stress, and  $k$  is the yield stress in pure shear. For an elastic perfectly plastic solid the total strain rates are defined by

$$\partial_t \epsilon_{ij} = \partial_t \epsilon_{ij}^{el} + \partial_t \epsilon_{ij}^{pl} \quad (1.3)$$

Here the elastic strain-rates are defined by

$$\partial_t \epsilon_{ij}^{el} = \frac{1}{2\mu} \left( \partial_t \sigma_{ij} - \frac{\nu}{1-\nu} \partial_t \sigma_{kk} \delta_{ij} \right) \quad (1.4)$$

while the plastic strain rates are

$$\partial_t \epsilon_{ij}^{pl} = \lambda s_{ij} \quad (1.5)$$

Equations (1.3) - (1.5) are the Prandtl-Reuss flow equations



In (1.4)  $\mu$  and  $\nu$  are the elastic shear modulus and Poisson's ratio, respectively, while in (1.5)  $s_{ij}$  defines the stress deviator, and  $\lambda$  is a non-negative proportionality factor, which may vary in space and time. Equations (1.1) - (1.5) should be supplemented by appropriate boundary conditions.

The speed of the crack tip is  $v = d\ell(t)/dt$ , where  $\ell(t)$  is the monotonically increasing crack length, and where  $v(t)$  and  $dv/dt$  are continuous. The moving and the stationary coordinate system are related by

$$x = x_1 - \ell(t) \quad (1.6)$$

By means of (1.6) the material time derivatives are transformed to the moving coordinate system by the relations

$$\partial_t(\quad) = \frac{\partial}{\partial t} - v(t) \frac{\partial}{\partial x} \quad (1.7)$$

$$\partial_t^2(\quad) = \frac{\partial^2}{\partial t^2} - \dot{v}(t) \frac{\partial}{\partial x} - 2v(t) \frac{\partial^2}{\partial t \partial x} + v^2(t) \frac{\partial^2}{\partial x^2} \quad (1.8)$$

For the case that  $v(t)$  approaches a constant value  $v_\infty$  as  $t$  increases, it may be assumed that steady-state fields of stress and deformation are established relative to the coordinate system moving with the crack tip. This assumption implies that the time derivatives on the right-hand sides of (1.7) and (1.8) may be neglected, which leads to the usual results that

$$\partial_t(\quad) \sim -v_\infty \frac{\partial}{\partial x}; \quad \partial_t^2(\quad) \sim v_\infty^2 \frac{\partial^2}{\partial x^2} \quad (1.9a,b)$$

In the sequel the speed of the crack tip will appear in the "Mach number"  $M$ , which is defined as

$$M = v_{\infty} / (\mu/\rho)^{\frac{1}{2}} \quad (1.10)$$

where  $(\mu/\rho)^{\frac{1}{2}}$  defines the velocity of shear waves in an elastic solid with shear modulus  $\mu$ .

## 2. Crack-Propagation in Anti-Plane Strain

In the moving coordinate system  $(x, y, z)$ , steady-state motion in anti-plane strain is defined by a displacement  $w(x, y)$  in the  $z$ -direction. By using (1.9b), the equation of motion (1.1) then reduces to

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} - \rho v_{\infty}^2 \partial_{xx} w = 0 \quad (2.1)$$

The Tresca yield condition (1.2) may be expressed in the form

$$\sigma_{xz}^2 + \sigma_{yz}^2 \leq k^2, \quad (2.2)$$

and the Prandtl-Reuss flow law (1.3) yields by using (1.9a)

$$\partial_{xx} w = \frac{1}{\mu} \partial_x \sigma_{xz} - 2\lambda \sigma_{xz} \quad (2.3)$$

$$\partial_{xy} w = \frac{1}{\mu} \partial_x \sigma_{yz} - 2\lambda \sigma_{yz} \quad (2.4)$$

It is sufficient to consider the solution in the half-plane  $y \geq 0$ .

The boundary conditions at  $y = 0$  are

$$x < 0 : \quad \sigma_{yz} = 0 \quad (2.5)$$

$$x \geq 0 : \quad \sigma_{yz} > 0, \quad w = 0 \quad (2.6a, b)$$

The yield condition (2.2) is identically satisfied by

$$\sigma_{xz} = -k \sin w, \quad \sigma_{yz} = k \cos w \quad (2.7a, b)$$

By introducing the strain component

$$w_x = \partial_x w \quad (2.8)$$



together with (2.7a,b) in (2.1), and eliminating  $2\lambda$  from (2.3) and (2.4), we obtain

$$\cos w \frac{\partial w}{\partial x} + \sin w \frac{\partial w}{\partial y} + M^2 \frac{\mu}{k} \frac{\partial w}{\partial x} = 0 \quad (2.9)$$

and

$$\cos w \frac{\partial w}{\partial x} + \sin w \frac{\partial w}{\partial y} + \frac{k}{\mu} \frac{\partial w}{\partial x} = 0 \quad (2.10)$$

where  $M$  is defined by (1.10).

Equations (2.9) and (2.10) constitute a hyperbolic system of equations. A brief study of (2.9) and (2.10), including a discussion of the characteristic curves and the corresponding Riemann invariants has been presented by Achenbach, Burgers and Dunayevsky (1979). These authors showed that a solution of (2.9)-(2.10) satisfying the boundary conditions (2.5) and (2.6a,b) can be constructed by using only simple and uniform fields. In the present paper we start off with so-called "simple wave" solutions, which are solutions for which the dependent variables are functions of each other. A brief discussion of simple wave solutions is presented in Appendix A.

It is easily verified that (2.9) and (2.10) are of the general form given by (A.1), where

$$u_1 = w, \quad \text{and} \quad u_2 = w_x \quad (2.11a,b)$$

while the components of the matrix  $L$  are defined by

$$L = \begin{bmatrix} \cot w & \frac{\mu}{k} \frac{M^2}{\sin w} \\ \frac{k}{\mu} \frac{1}{\sin w} & \cot w \end{bmatrix} \quad (2.12)$$

The characteristic directions follow from (A.7) as

$$\gamma_{\pm} = \frac{dx}{dy} = \cot \omega \pm \frac{M}{\sin \omega} \quad (2.13)$$

It follows from (A.8) that

$$\frac{d\omega}{\cot \omega - \gamma} = - \frac{dw_x}{k/(\mu \sin \omega)} \quad (2.14)$$

Substitution of (2.13) into (2.14) yields

$$\text{for } \gamma_{+}: \quad dw_x = (k/\mu M) d\omega \quad (2.15)$$

$$\text{for } \gamma_{-}: \quad dw_x = - (k/\mu M) d\omega \quad (2.16)$$

These relations can easily be integrated to yield

$$\text{for } \gamma_{+}: \quad w_x = (k/\mu M) \omega + w_x^{+} \quad (2.17)$$

$$\text{for } \gamma_{-}: \quad w_x = - (k/\mu M) \omega + w_x^{-} \quad (2.18)$$

where  $w_x^{+}$  and  $w_x^{-}$  are constants of integration.

Equation (2.17) leads to a negative value of  $\lambda$ . Hence we use only the relation given by (2.18). Substitution of (2.18) into either (2.9) or (2.10) gives

$$(\cos \omega - M) \partial_x \omega + \sin \omega \partial_y \omega = 0 \quad (2.19)$$

Along the family of characteristic curves defined by  $\gamma_{-}$ , see (2.13), (2.19) reduces to  $d\omega = 0$ , i.e.,  $\omega = \text{constant}$ . Consequently the relations given by (2.13) for the  $\gamma_{-}$  family can be integrated to yield

$$y - \frac{\sin \omega}{\cos \omega - M} x = \varphi(\theta) \quad (2.20)$$

where  $\theta$  is the polar angle defined by

$$\tan \theta = y/x, \quad (2.21)$$

and  $\varphi(\theta)$  is an arbitrary function. However, by virtue of the boundary

condition (2.6b) we must have  $\varphi(\theta) \equiv 0$ . Thus

$$y = \frac{\sin w}{\cos w - M} x \quad (2.22)$$

Equations (2.22) and (2.21) imply a relation between  $w$  and  $\theta$ , which can be solved to yield

$$\cos w = M \sin^2 \theta + (1-M^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta \quad (2.23)$$

The stresses then follow from (2.7a,b) as

$$\sigma_{xz} = -k [(1-M^2 \sin^2 \theta)^{\frac{1}{2}} - M \cos \theta] \sin \theta \quad (2.24)$$

$$\sigma_{yz} = k [(1-M^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta + M \sin^2 \theta] \quad (2.25)$$

Substitution of (2.23) into (2.18) yields

$$w_x = - (k/\mu M) \cos^{-1} [M \sin^2 \theta + (1-M^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta] \quad (2.26)$$

where we have taken  $w_x^- \equiv 0$  to satisfy the boundary condition (2.6b).

From (2.10) we obtain

$$\partial_y w_x \equiv \partial_x w_y = \frac{k}{\mu M} \frac{\cos w}{\sin w} \partial_x w - \frac{k}{\mu} \frac{1}{\sin w} \partial_x w \quad (2.27)$$

where (2.18) has also been used, and  $w_y$  is defined as

$$w_y = \partial w / \partial y \quad (2.28)$$

Equation (2.27) can be integrated with respect to  $x$ , to yield

$$\begin{aligned} w_y = \frac{k}{\mu} \left\{ \frac{1-M}{2M} \ln[1-M \sin^2 \theta - (1-M^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta] \right. \\ \left. + \frac{1+M}{2M} \ln[1+M \sin^2 \theta + (1-M^2 \sin^2 \theta)^{\frac{1}{2}} \cos \theta] \right\} + \psi(y) \end{aligned} \quad (2.29)$$

where  $\psi(y)$  is an as yet undetermined function.

The function  $\lambda(r, \theta)$  can be obtained from (2.24) - (2.26) and either (2.3) or (2.4). After some manipulation we find

$$\mu r \lambda \sim \frac{1}{2M} (1 - M^2 \sin^2 \theta)^{\frac{1}{2}} > 0 \quad (2.30)$$

which satisfies the essential condition on  $\lambda$ .

It is noted that the expression for  $\sigma_{yz}$  given by (2.25) cannot satisfy the condition that  $\sigma_{yz} = 0$  at  $\theta = \pi$ , as required by (2.5). This suggests that (2.23) - (2.30) are valid only in a domain  $0 \leq \theta \leq \theta^*$ , while another solution holds for  $\theta^* \leq \theta \leq \pi$ . This other solution cannot be a simple wave solution, since no other than the one given by (2.23) - (2.30) is available. A suitable solution for  $\theta^* \leq \theta \leq \pi$  is, however, given by a uniform field of the form:

$$\omega = \text{constant}, \quad w_x = \text{constant} \quad (2.31)$$

Such a field clearly satisfies (2.9) and (2.10). An appropriate choice for  $\omega$  is

$$\omega = \pi/2, \quad (2.32)$$

which, by virtue of (2.7a,b), leads to the following stresses for  $\theta^* \leq \theta \leq \pi$

$$\sigma_{xz} = -k, \quad \sigma_{yz} = 0 \quad (2.33a,b)$$

Thus, the boundary condition (1.5) is satisfied. It immediately follows from (2.23) and (2.32) that the angle  $\theta^*$  must satisfy the equation

$$M \sin^2 \theta^* + (1 - M^2 \sin^2 \theta^*)^{\frac{1}{2}} \cos \theta^* = 0 \quad (2.34)$$

Hence

$$\theta^* = -\tan^{-1} (1/M) \quad (2.35)$$

Thus  $\theta^*$  varies from  $\theta^* = 90^\circ$  at  $M = 0$  to  $\theta^* = 135^\circ$  at  $M = 1$ .

Substitution of (2.35) in (2.26) and (2.29), and the requirement



that the strains should be continuous at  $\theta = \theta^*$  leads to the following results for  $\theta^* \leq \theta \leq \pi$ :

$$w_x = - \pi k / 2 \mu M \quad (2.36)$$

$$w_y = \psi(y) \quad (2.37)$$

Finally, an approximate expression for  $\psi(y)$  can be found from the condition that  $w_y$  should be bounded for  $y = 0$ ,  $x \sim r_p$ , where  $r_p$  is the length of the plastic zone. For small  $y$  the expression for  $w_y$  given by (2.29) then implies that

$$\psi(y) = \frac{k}{\mu} \left[ \frac{1-M}{M} \ln(r_p/|y|) - \frac{1-M}{M} \ln(1-M) - \ln 2 + 1 \right] \quad (2.38)$$

The results presented in this Section are essentially the same as those found by Slepyan (1976) who used an approach in which only the most singular terms,  $O(1/r)$ , were retained in the governing equations.

For three values of  $M$  the stress components  $\sigma_{xz}$ , given by (2.24), (2.33a), and  $\sigma_{yz}$  given by (2.25), (2.33b), have been plotted in Figs. 2 and 3, respectively. The curves show a substantial dependence of the stresses on  $M$ . For  $M = 0$ , (2.24) and (2.25) reduce to  $\sigma_{xz} = -k \sin \theta$  and  $\sigma_{yz} = k \cos \theta$ , respectively, which are the expressions for the quasi-static stresses derived by Chitaley and McClintock (1971). Thus, as one would perhaps expect intuitively, the dynamic stresses reduce to the quasi-static ones in the limit  $M \rightarrow 0$ . This is, however, not true for the strains as discussed later.

For  $0 \leq \theta \leq \theta^*$ , the strain  $(\mu/k)w_x$ , which is given by (2.26), has been plotted in Fig. 4. Again, a very noticeable dependence on  $M$  is observed.

It follows from (2.29) and (2.38) that  $w_y$  is singular at the crack tip. In the domain  $0 \leq \theta \leq \theta^*$  we find

$$w_y \sim \frac{k}{\mu} \frac{1-M}{M} \ln(r/r_p) \quad (2.39)$$

This result shows that  $w_y$  not only becomes unbounded as  $r \rightarrow 0$ , but also as  $M \rightarrow 0$ . The strain component  $w_x$  is bounded as  $r \rightarrow 0$ , but it becomes unbounded in the limit  $M \rightarrow 0$ . Thus, for the strains there is no uniform transition from dynamic to quasi-static conditions.

In the loading zone the quasi-static solution is of the form  $w_x \sim \ln(r/r_p) \sin \theta$  as  $r \rightarrow 0$ . Behind the crack tip the quasi-static solution shows a region of elastic unloading. As a material point passes the crack tip, the increments of  $w_x$  change sign, and for the quasi-static case this can only be achieved by elastic unloading. For the asymptotically valid dynamic solution presented in this paper the loading zone extends, however, completely around the crack tip, even though different expressions hold for  $0 \leq \theta \leq \theta^*$  and  $\theta^* \leq \theta \leq \pi$ , respectively. It is speculated that the length dimensions of the zone in which the present dynamic results are valid are of order  $O(M^2)$ . Outside of this zone another (dynamic) solution which does describe unloading and which does show a uniform transition to the quasi-static solution may occur. As  $M \rightarrow 0$  the dynamic zone then shrinks on the crack tip, and in the limit  $M \equiv 0$  the zone has vanished altogether, and the quasi-static solution describes the near-tip strains.

A final result of interest is the crack-opening angle. This angle, which is defined by  $\tan \frac{1}{2} \alpha = |w_x|$  at  $\theta = \pi$ , follows from (2.36) as

$$\tan \alpha = 2 \tan^{-1} (\pi k / 2 \mu M) \quad (2.40)$$

As  $M \rightarrow 0$  we find  $\alpha = \pi$ , which is in agreement with the quasi-static result.



### 3. Crack-Propagation in Plane Strain

In the moving coordinate system  $(x, y, z)$ , steady-state motion in plane strain is defined by displacements  $u(x, y)$  and  $v(x, y)$  in the  $x$ - and  $y$ -directions, respectively. We assume that  $\sigma_z$  is the intermediate stress, i.e.,  $\sigma_x \leq \sigma_z \leq \sigma_y$ . For the elastic case this is true, for the elastic-plastic problem it must be checked a posteriori. It was noted by Koiter (1953), see also Geiringer (1973, p. 514), that the Tresca yield condition then implies that

$$\partial_t \epsilon_z^{pl} = \epsilon_z^{pl} \equiv 0 \quad (3.1)$$

The condition of plane strain consequently reduces to  $\epsilon_z^{el} = 0$ , which by virtue of (1.4) yields the result

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad (3.2)$$

We now introduce the following new variables

$$\sigma_- = \frac{1}{2} (\sigma_x - \sigma_y) \quad (3.3)$$

$$\sigma = \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \quad (3.4)$$

$$u_x = \partial_x u \quad (3.5)$$

$$v_x = \partial_x v \quad (3.6)$$

By the use of Eq.(3.2) it is then easily checked that the equations of motion in the moving coordinate system may be expressed in the form

$$\partial_x \sigma_- + \frac{3}{2(1+\nu)} \partial_x \sigma + \partial_y \sigma_{xy} - \rho v_\infty^2 \partial_x u_x = 0 \quad (3.7)$$

$$-\partial_y \sigma_- + \frac{3}{2(1+\nu)} \partial_y \sigma + \partial_x \sigma_{xy} - \rho v_\infty^2 \partial_x v_x = 0 \quad (3.8)$$

Expressions for  $\partial_x \epsilon_x$  and  $\partial_x \epsilon_y$  follow from the Prandtl-Reuss flow equa-

tions (1.3). By considering  $\partial_x(\epsilon_x - \epsilon_y)$  we find

$$\partial_y v_x - \partial_x u_x = (2\lambda/v_\infty) \sigma_- - (1/\mu) \partial_x \sigma_- \quad (3.9)$$

Similarly the relation for  $\partial_x \epsilon_{xy}$  yields

$$\partial_x v_x + \partial_y u_x = - (2\lambda/v_\infty) \sigma_{xy} + (1/\mu) \partial_x \sigma_{xy} \quad (3.10)$$

For plane strain the Tresca yield condition can be expressed in the form

$$\sigma_-^2 + \sigma_{xy}^2 = k^2 \quad (3.11)$$

Finally, the boundary conditions at  $y = 0$  are

$$x < 0 : \quad \sigma_y = 0, \quad \sigma_{xy} = 0 \quad (3.12)$$

$$x \geq 0 : \quad v = 0, \quad \sigma_{xy} = 0 \quad (3.13)$$

It is noted that the yield condition (3.11) is satisfied identically

by

$$\sigma_- = -k \cos w \quad (3.14)$$

$$\sigma_{xy} = -k \sin w \quad (3.15)$$

By introducing these expressions in (3.7) - (3.10), and subsequently eliminating  $(\lambda/v_\infty)$  we arrive at a system of equations of the general form

$$\sum_{j=1}^4 L_{ij} \frac{\partial u_j}{\partial x} + \frac{\partial u_i}{\partial y} = 0, \quad (3.16)$$

where

$$u_1 = w, \quad u_2 = \sigma, \quad u_3 = u_x, \quad u_4 = v_x, \quad (3.17)$$

and the matrix  $L$  with components  $L_{ij}$  is defined by

$$L = \begin{bmatrix} -\tan w & -\frac{3}{2k(1+\nu)\cos w} & \frac{\mu}{k} \frac{M^2}{\cos w} & 0 \\ \frac{2k(1+\nu)}{3 \cos w} & -\tan w & \frac{2}{3}\mu(1+\nu)M^2 \tan w & -\frac{2}{3}\mu(1+\nu)M^2 \\ \frac{k}{\mu} \frac{1}{\cos w} & \frac{3(1-2\nu)}{2\mu(1+\nu)} \tan w & -2 \tan w & 1 \\ 0 & -\frac{3(1-2\nu)}{2\mu(1+\nu)} & 1 & 0 \end{bmatrix} \quad (3.18)$$

Equation (3.16) is of the general form stated by Eq.(A.1) of Appendix A. Just as for the anti-plane case we seek simple wave solutions of Eq. (3.16). Unfortunately, for the plane strain case the system is too complicated to yield an analytical solution. It is, however, possible to obtain simple wave solutions for small values of  $M$ , where  $M$  is defined by (1.10), by using a perturbation method which is discussed in some detail in Appendix B.

Expressions for the characteristic directions  $\gamma = dx/dy$  at small values of  $M$  are given by (B.17) and (B.18). The relevant characteristic direction for  $y \geq 0$  is

$$\gamma_3^{(1)} = \frac{1-\sin w}{\cos w} - M \left( \frac{1-\nu}{1+\sin w} \right)^{\frac{1}{2}} + O(M^2) \quad (3.19)$$

where  $w$  depends analytically on  $M$ .

As discussed in Appendix B, the following relations on the characteristic curves defined by  $\gamma_3^{(1)}$  are obtained:

$$\sigma = -\frac{2}{3}k(1+\nu)w + \sigma_0 + O(M^2) \quad (3.20)$$

$$u_x = -\frac{k}{\mu} \left[ \frac{2}{M}(1-\nu)^{\frac{1}{2}}(1+\sin w)^{\frac{1}{2}} - \nu w \right] + u_x^0 + O(M^2) \quad (3.21)$$

$$v_x = \frac{2k}{\mu M} (1-\nu)^{\frac{1}{2}} (1-\sin\omega)^{\frac{1}{2}} + v_x^0 + O(M^2) \quad (3.22)$$

Substitution of (3.20)-(3.22) into the first of (3.16) yields

$$\left[ \frac{1-\sin\omega}{\cos\omega} - M \left( \frac{1-\nu}{1+\sin\omega} \right)^{\frac{1}{2}} \right] \partial_x \omega + \partial_y \omega = 0 \quad (3.23)$$

Along the family of characteristic curves defined by  $v_3^{(1)}$ , (3.23)

reduces to  $d\omega = 0$ , i.e.,  $\omega = \text{constant}$ . Consequently the relation (3.19)

given by  $v_3^{(1)} = dx/dy$ , can be integrated to yield

$$x - \left[ \frac{1-\sin\omega}{\cos\omega} - M \left( \frac{1-\nu}{1+\sin\omega} \right)^{\frac{1}{2}} \right] y = \varphi(\theta) \quad (3.24)$$

where  $\varphi(\theta)$  is an arbitrary function of the polar angle  $\theta$ . For the application at hand the appropriate choice is  $\varphi(\theta) \equiv 0$  for a central field of characteristics.

Equations (3.24) and (2.21) now yield the following relations

$$\sin\omega = -\cos 2\theta - M [2(1-\nu)]^{\frac{1}{2}} \sin\theta \sin 2\theta + O(M^2) \quad (3.25)$$

$$\cos\omega = \sin 2\theta - M [2(1-\nu)]^{\frac{1}{2}} \sin\theta \cos 2\theta + O(M^2) \quad (3.26)$$

and hence

$$\omega = 2\theta - \frac{\pi}{2} - M [2(1-\nu)]^{\frac{1}{2}} \sin\theta + O(M^2) \quad (3.27)$$

Substitution of this result into (3.14) and (3.15) yields

$$\sigma_{xy} = k \{ \cos 2\theta + M [2(1-\nu)]^{\frac{1}{2}} \sin\theta \sin 2\theta \} \quad (3.28)$$

$$\sigma_{\perp} = k \{ -\sin 2\theta + M [2(1-\nu)]^{\frac{1}{2}} \sin\theta \cos 2\theta \} \quad (3.29)$$

The average pressure follows from (3.21) as

$$\sigma = \sigma_0 - \frac{2}{3}k(1+\nu) \left\{ 2\theta - \frac{\pi}{2} - M [2(1-\nu)]^{\frac{1}{2}} \sin\theta \right\} + O(M^2) \quad (3.30)$$



By combining (3.2), (3.3) and (3.4) we find

$$\begin{aligned}\sigma_x &= \sigma_- + \frac{3\sigma}{2(1+\nu)} \\ &= \frac{3\sigma_o}{2(1+\nu)} + k \left( \frac{\pi}{2} - 2\theta - \sin 2\theta \right) + Mk [2(1-\nu)]^{\frac{1}{2}} \cos \theta \sin 2\theta + O(M^2)\end{aligned}\quad (3.31)$$

$$\begin{aligned}\sigma_y &= \frac{3\sigma}{2(1+\nu)} - \sigma_- \\ &= \frac{3\sigma_o}{2(1+\nu)} + k \left( \frac{\pi}{2} - 2\theta + \sin 2\theta \right) + 2Mk [2(1-\nu)]^{\frac{1}{2}} \sin^3 \theta + O(M^2)\end{aligned}\quad (3.32)$$

$$\sigma_z = \frac{3\nu}{1+\nu} \sigma_o - 2\nu k \left\{ 2\theta - \frac{\pi}{2} - M[2(1-\nu)]^{\frac{1}{2}} \sin \theta \right\} + O(M^2)\quad (3.33)$$

In these expressions the constant  $\sigma_o$  is still undetermined. Additional relations can, however, be obtained from the boundary conditions (3.12) and (3.13). Clearly (3.28) and (3.32) can neither satisfy the conditions stated by (3.12) nor by (3.13) which suggests that (3.28) - (3.33) are valid only in a domain  $\theta_1^* \leq \theta \leq \theta_2^*$ , while other solutions hold for  $0 \leq \theta \leq \theta_1^*$  and  $\theta_2^* \leq \theta \leq \pi$ . Appropriate solution for the latter domains are again constant states.

For both  $0 \leq \theta \leq \theta_1^*$  and  $\theta_2^* \leq \theta \leq \pi$  constant states imply  $\sigma_{xy} = 0$ . Equation (3.28) and the condition that  $\sigma_{xy}$  should be continuous at  $\theta = \theta_1^*$  and  $\theta = \theta_2^*$  then yields the equation

$$\cos 2\theta^* + M [2(1-\nu)]^{\frac{1}{2}} \sin \theta^* \sin 2\theta^* = 0\quad (3.34)$$

which to first order in  $M$  has the solutions

$$\theta_1^* = \frac{\pi}{4} + \frac{1}{2}(1-\nu)^{\frac{1}{2}} M + O(M^2)\quad (3.35)$$

$$\theta_2^* = \frac{3\pi}{4} + \frac{1}{2}(1-\nu)^{\frac{1}{2}} M + O(M^2)\quad (3.36)$$

On the basis of Eq.(3.32), the condition that  $\sigma_y$  should vanish at  $\theta = \theta_2^*$  yields

$$\frac{3\sigma_o}{2(1+\nu)} = (1+\pi)k \quad (3.37)$$

The complete stress fields in the constant state regions then are easily computed. In summary:

$$0 \leq \theta \leq \theta_1^* :$$

$$\sigma_{xy} = 0 \quad (3.38)$$

$$\sigma_y = k(2+\pi) + O(M^2) > 0 \quad (3.39)$$

$$\sigma_x = k\pi + O(M^2) \quad (3.40)$$

$$\sigma_z = 2\nu k(1+\pi) \quad (3.41)$$

$$\theta_1^* \leq \theta \leq \theta_2^* :$$

$$\sigma_{xy} = 0 ; \sigma_y = 0 ; \sigma_x = 2k ; \sigma_z = 2\nu k \quad (3.42a,b,c,d)$$

We now proceed with the computation of the strains. In the sector  $\theta_1^* \leq \theta \leq \theta_2^*$  we obtain by substituting (3.27) into (3.21) and (3.22)

$$\epsilon_x = u_x = u_x^o + \nu \frac{k}{\mu} (2\theta - \frac{\pi}{2}) + \frac{k}{\mu} (1-\nu) \sin 2\theta - \frac{1}{M} \frac{2k}{\mu} [2(1-\nu)]^{\frac{1}{2}} \sin \theta \quad (3.43)$$

$$v_x = v_x^o + 2\frac{k}{\mu} (1-\nu) \sin^2 \theta + \frac{1}{M} \frac{2k}{\mu} [2(1-\nu)]^{\frac{1}{2}} \cos \theta \quad (3.44)$$

By using the relation

$$\sigma = \frac{2(1+\nu)\mu}{3(1-2\nu)} (\epsilon_x + \epsilon_y) \quad (3.45)$$

in conjunction with (3.30) and (3.43),  $\epsilon_y$  follows as

$$\begin{aligned} \epsilon_y = & -u_x^o + \frac{k}{\mu} (1-2\nu)(1+\pi) - \frac{k}{\mu} (1-\nu) (2\theta - \frac{\pi}{2}) + \frac{1}{M} \frac{2k}{\mu} [2(1-\nu)]^{\frac{1}{2}} \sin \theta \\ & - \frac{k}{\mu} (1-\nu) \sin 2\theta \end{aligned} \quad (3.46)$$



Next, consider (3.16) for  $i = 3$ , and eliminate  $\partial\sigma/\partial x$ ,  $\partial u_x/\partial x$  and  $\partial v_x/\partial x$  by using (3.20) - (3.22). The result is

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = f(\omega) \frac{\partial \omega}{\partial x} \quad (3.47)$$

where  $f(\omega)$  can easily be determined. Equation (3.47) can now be integrated with respect to  $x$ , and  $\omega$  can subsequently be eliminated by using (3.27).

The result of these manipulations is

$$u_y = -\frac{k}{\mu} \ln(2\sin^2\theta) - \frac{2k}{\mu}(1-\nu) \cos^2\theta + \frac{1}{M} \frac{2k}{\mu} [2(1-\nu)]^{\frac{1}{2}} [\cos\theta + \ln(\tan \frac{\theta}{2})] + 2\psi(y) \quad (3.48)$$

Here  $\psi(0) = 0$ , but otherwise  $\psi(y)$  is an arbitrary function. The strain component  $\epsilon_{xy}$  then follows from (3.44) and (3.47) as

$$\epsilon_{xy} = -\frac{k}{\mu}(1-\nu) \cos 2\theta - \frac{k}{2\mu} \ln(2\sin^2\theta) + \frac{1}{M} \frac{k}{\mu} [2(1-\nu)]^{\frac{1}{2}} [2\cos\theta + \ln(\tan \frac{\theta}{2})] + \frac{1}{2} u_x^o + \psi(y) \quad (3.49)$$

At  $\theta = \theta_1^*$ , Eq.(3.44) yields

$$v_x = v_x^o + \frac{1}{M} \frac{2k}{\mu}(1-\nu)^{\frac{1}{2}} + O(M) \quad (3.50)$$

Since  $v_x$  is continuous at  $\theta = \theta_1^*$ , and uniform in the domain  $0 \leq \theta \leq \theta_1^*$ , and since by symmetry considerations  $v_x$  must vanish at  $\theta = 0$ , it follows that

$$v_x^o = -\frac{1}{M} \frac{2k}{\mu}(1-\nu)^{\frac{1}{2}} + O(M) \quad (3.51)$$

By evaluating (3.43), (3.46) and (3.49) at  $\theta = \theta_1^*$  we then obtain for  $0 \leq \theta \leq \theta_1^*$ :

$$\epsilon_x = u_x^o - \frac{1}{M} \frac{2k}{\mu}(1-\nu)^{\frac{1}{2}} \quad (3.52)$$

$$\epsilon_y = -u_x^0 + \frac{k}{\mu}(1-2\nu)(1+\pi) + \frac{1}{M} \frac{2k}{\mu}(1-\nu)^{\frac{1}{2}} \quad (3.53)$$

$$\epsilon_{xy} = \psi(y) \quad (3.54)$$

A computation of  $u_x^0$  can be based on the assumption that at  $\theta = 0$  the domain of validity of (3.52) - (3.54) borders directly on the zone of elastic deformation, and that  $\epsilon_x$ ,  $\epsilon_y$  and  $\sigma_x$  are continuous at the elastic-plastic boundary. If in the plastic zone  $\sigma_x$  is represented by (3.40), and outside the plastic zone  $\sigma_x$  is computed by using Hooke's law with  $\epsilon_x$  and  $\epsilon_y$  as defined by (3.52) and (3.53), then continuity of  $\sigma_x$  yields

$$u_x^0 = \frac{1}{M} \frac{2k}{\mu}(1-\nu)^{\frac{1}{2}} + \frac{k}{\mu} \left[ \frac{\pi}{2} - \nu(1+\pi) \right] \quad (3.55)$$

The strains in the domain  $0 \leq \theta \leq \theta^*$  then reduced to

$$\epsilon_x = \frac{k}{\mu} \left[ \frac{\pi}{2} - \nu(1+\pi) \right] \quad (3.56)$$

$$\epsilon_y = \frac{k}{\mu} \left[ -\frac{\pi}{2} + (1-\nu)(1+\pi) \right] \quad (3.57)$$

By evaluating (3.43), (3.46) and (3.49) at  $\theta = \theta_2^*$ , and using (3.51) and (3.55) we find for  $\theta_2^* \leq \theta \leq \pi$ :

$$\epsilon_x = \frac{k}{\mu} \left[ \frac{\pi}{2} - 1 + (1-\nu)/\sqrt{2} \right] \quad (3.58)$$

$$\epsilon_y = \frac{k}{\mu} \left[ -\frac{\pi}{2} + (1-\nu)(4-\sqrt{2})/2 \right] \quad (3.59)$$

$$\epsilon_{xy} = -\frac{1}{M} \frac{3k}{\mu}(1-\nu)^{\frac{1}{2}} - \frac{k}{\mu}(1-\nu)(1-1/\sqrt{2}) + \frac{k}{\mu} [2(1-\nu)]^{\frac{1}{2}} \ln(\tan 3\pi/8) + \psi(y) \quad (3.60)$$

The crack opening angle, which is defined by  $\tan \alpha = |\nu_x|$ , follows from (3.44) by substituting  $\theta = \theta_2^*$ . The result is

$$\tan \alpha = \frac{1}{M} \frac{4k}{\mu}(1-\nu)^{\frac{1}{2}} \quad (3.61)$$

The expressions for the stresses given by (3.31)-(3.33) show only a weak dependence on  $M$ . In the limit of vanishing crack-tip speed ( $M \rightarrow 0$ ) these expressions reduce to the corresponding quasi-static results as, for example, presented by Rice and Tracey (1973). In the range  $\theta_1^* < \theta < \theta_2^*$ , the strains, given by (3.43), (3.46) and (3.49) strongly depend on  $M$  due to the presence of terms of order  $O(M^{-1})$ . The strain  $\epsilon_x$  and  $\epsilon_y$  have been plotted in Figs. 5 and 6 for  $M = 0.1$  and  $M = 0.01$ . In the limit  $M \rightarrow 0$  the strains become unbounded.

The results presented in this section do not show elastic unloading. For the quasi-static problem it has been shown by Rice, Drugan and Sham (1979) that an elastic unloading zone must exist in the immediate vicinity of the crack tip.

Thus, the near-tip in-plane asymptotic results presented in this Section, which take into account dynamic effects, display the same anomalies that were observed in Section 2 for the case of anti-plane strain. There is no uniform transition from the dynamic to the quasi-static results as the crack-tip speed becomes very small, and the dynamic solutions do not show elastic unloading in the immediate vicinity of the crack tip. Just as for the Mode-III case it is speculated that the actual transition from dynamic to quasi-static results is achieved because the zone of validity of the results of this section shrinks on the crack tip in the limit  $M \rightarrow 0$ .

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## APPENDIX A

Simple Wave Solutions

The systems of equations that appear in this paper are of the general form

$$\sum_{j=1}^n L_{ij} \frac{\partial u_j}{\partial x} + \frac{\partial u_i}{\partial y} = 0 \quad i = 1, 2, \dots, n \quad (A.1)$$

Equation (A.1) defines a system of  $n$  simultaneous first-order partial differential equations for the dependent variables  $u_1, u_2, \dots, u_n$ . The equations are homogeneous, and  $L_{ij}$  are functions of  $x, y$  and  $u_i$ , but not of the derivatives of the  $u_i$ . Such equations are called quasi-linear.

As discussed by e.g. Bland (1969) solutions of Eq.(A.1) are simple wave solutions if all the  $u_i$  are functions of one another, or equivalently, if all  $u_i$  are functions of another variable, say  $\varphi$ . For  $u_i = u_i(\varphi)$  we have

$$\frac{\partial u_i}{\partial x} = u'_i \frac{\partial \varphi}{\partial x}, \quad \frac{\partial u_i}{\partial y} = u'_i \frac{\partial \varphi}{\partial y}, \quad \text{where } u'_i = \frac{du_i}{d\varphi} \quad (A.2)$$

Equation (A.1) can then be rewritten as

$$\left( \sum_{j=1}^n L_{ij} u'_j \right) \frac{\partial \varphi}{\partial x} + u'_i \frac{\partial \varphi}{\partial y} = 0 \quad (A.3)$$

Along any curve in the  $xy$ -plane we have

$$dx \frac{\partial \varphi}{\partial x} + dy \frac{\partial \varphi}{\partial y} = d\varphi \quad (A.4)$$

Hence Eq.(A.3) define total differentials along curves defined by

$$\gamma = \frac{dx}{dy} = \frac{\sum_{j=1}^n L_{ij} u'_j}{u'_i} = \frac{\sum_{j=1}^n L_{ij} du_j}{du_i} \quad (A.5)$$

Equations (A.3) are true for all values of  $i$ , and therefore, the right-

hand side of Eq.(A.5) is the same for all values of  $i$ . Hence

$$\sum_{j=1}^n (L_{ij} - \gamma \delta_{ij}) du_j = 0 \quad i = 1, 2, \dots, n \quad (\text{A.6})$$

This system of equations has non-trivial solutions only if the determinant of the coefficients vanishes, i.e., if

$$|L_{ij} - \gamma \delta_{ij}| = 0 \quad (\text{A.7})$$

Equation (A.7) is recognized as the equation for the slopes  $\gamma = dx/dy$  of the characteristics.

The relations holding on a characteristic of a simple wave follow from Eqs.(A.6). Since the  $du_j$  are the right eigenvectors of  $L_{ij} - \gamma \delta_{ij}$ , they are proportional to the cofactors of any row of  $L_{ij} - \gamma \delta_{ij}$ , provided that such cofactors are not all zero. Thus we have

$$\frac{du_1}{C_1} = \frac{du_2}{C_2} = \dots = \frac{du_n}{C_n} \quad (\text{A.8})$$

where  $C_i$ ,  $i = 1, 2, \dots, n$ , are such cofactors.



## APPENDIX B

Perturbation Method

The system of governing equations for the case of plane strain, given by (3.16), is of the general form (A.1). We again seek simple wave solutions. Formally the equation for the characteristic curves is given by Eq.(A.7). This equation is, however, much too complicated to be solved analytically. Thus, it was attempted to obtain simple wave solutions for small values of  $M$ , where  $M$  is defined by (1.20).

For  $M \equiv 0$ , the fourth-order equation given by (A.7), with  $L_{ij}$  defined by (3.18) yields the following double roots:

$$v_1^0 = - \frac{1 + \sin w^0}{\cos w^0} \quad (B.1)$$

$$v_2^0 = \frac{1 - \sin w^0}{\cos w^0} \quad (B.2)$$

Here the superscripts zero indicate that  $M \equiv 0$ . The multiplicity of the roots necessitates some special provisions before a perturbation procedure can be attempted. If expansions for small  $M$  were directly substituted in (A.7) the result would be a perturbation of (B.1) and (B.2), while the two other different roots would not be obtained. This difficulty can be circumvented by preceding the perturbation procedure by a transformation which would reduce  $L_{ij}$  at  $M = 0$  to the following canonical or Jordan form:

$$L_{ij}^0 = \begin{bmatrix} -\frac{1+\sin\omega^0}{\cos\omega^0} & 1 & 0 & 0 \\ 0 & -\frac{1+\sin\omega^0}{\sin\omega^0} & 0 & 0 \\ 0 & 0 & \frac{1-\sin\omega^0}{\sin\omega^0} & 1 \\ 0 & 0 & 0 & \frac{1-\sin\omega^0}{\sin\omega^0} \end{bmatrix} \quad (\text{B.3})$$

For the case of multiple eigenvalues formal methods are available to reduce a matrix to its canonical form. Here it suffices to state the transition to the Jordan representation can be achieved by introducing the new variables  $\hat{\omega}$ ,  $\hat{\sigma}$ ,  $\hat{u}_x$  and  $\hat{v}_x$  by the following substitutions

$$d\omega = -\frac{\mu}{k} \frac{1}{1-\nu} d\hat{\sigma} + \frac{\mu}{k} \frac{1}{1-\nu} d\hat{v}_x \quad (\text{B.4})$$

$$d\sigma = -\frac{2}{3} \mu \frac{1+\nu}{1-\nu} d\hat{\sigma} - \frac{2}{3} \mu \frac{1+\nu}{1-\nu} d\hat{v}_x \quad (\text{B.5})$$

$$du_x = -\frac{1+\sin\omega}{\cos\omega} d\hat{\omega} + \frac{\nu}{1-\nu} d\hat{\sigma} + \frac{1-\sin\omega}{\cos\omega} d\hat{u}_x + \frac{\nu}{1-\nu} d\hat{v}_x \quad (\text{B.6})$$

$$dv_x = d\hat{w} + d\hat{v}_x \quad (B.7)$$

For the purposes of the perturbation procedure we introduce in addition

$$d\tilde{v} = \frac{1}{M} d\hat{v}_x, \quad d\tilde{\sigma} = \frac{1}{M} d\hat{\sigma} \quad (B.8)$$

With the substitutions (B.4) - (B.8) the system of equations corresponding to (A.6) becomes

$$\begin{aligned} & - \left[ \gamma + \left(1 - \frac{\nu}{2} M^2\right) \frac{1+\sin w}{\cos w} \right] d\hat{w} + M \left(1 - \frac{\nu^2}{1-\nu} \frac{M^2}{2} \sin w\right) d\tilde{\sigma} + \\ & + \frac{\nu}{2} M^2 \frac{1-\sin w}{\cos w} d\hat{u}_x - \frac{\nu^2}{1-\nu} \frac{M^3}{2} \sin w d\tilde{v}_x = 0 \end{aligned} \quad (B.9)$$

$$M \frac{1-\nu}{1-\sin w} d\hat{w} - \left[ \gamma + \left(1 + \frac{\nu}{2} M^2\right) \frac{1+\sin w}{\cos w} \right] d\tilde{\sigma} - \frac{\nu}{2} M^2 \frac{1+\sin w}{\cos w} d\tilde{v}_x = 0 \quad (B.10)$$

$$\begin{aligned} & - \frac{\nu}{2} M^2 \frac{1+\sin w}{\cos w} d\hat{w} + \frac{\nu^2}{1-\nu} \frac{M^3}{2} \sin w d\tilde{\sigma} + \\ & + \left[ -\gamma + \frac{1-\sin w}{\cos w} \left(1 - \frac{\nu}{2} M^2\right) \right] d\tilde{u}_x + M \left(1 + \frac{\nu}{1-\nu} \frac{M^2}{2} \sin w\right) d\tilde{v}_x = 0 \end{aligned} \quad (B.11)$$

$$\frac{\nu}{2} M^2 \frac{1-\sin w}{\cos w} d\tilde{\sigma} + M \frac{1-\nu}{1+\sin w} d\tilde{u}_x + \left[ -\gamma + \frac{1-\sin w}{\cos w} \left(1 + \frac{\nu}{2} M^2\right) \right] d\tilde{v}_x = 0 \quad (B.12)$$

The canonical form  $L_{ij}^0$  given by (B.3) is recognized as appearing in the limit  $M = 0$ .

At this stage we can consider expansions in terms of powers of  $M$ :

$$\gamma = \gamma^0 + M \gamma^{(1)} \quad (B.13)$$

$$L_{ij} = L_{ij}^0 + M L_{ij}^{(1)} \quad (B.14)$$

together with analogous expansions for  $d\hat{w}$ ,  $d\tilde{\sigma}$ ,  $d\hat{u}_x$  and  $d\hat{v}_x$ , while  $w$  is assumed to depend analytically on  $M$ .

On the basis of Eq.(A.7) we obtain in the zero'th approximation the results given by (B.1) and (B.2). Considerations based on (A.8) yield the following relations along the characteristics

$$du_x = \pm \frac{1 - \sin \omega}{\cos \omega} dv_x, \quad d\omega = d\sigma = 0 \quad (B.15)$$

The first order approximations are obtained as

$$v_{1,2}^{(1)} = - \frac{1 + \sin \omega}{\cos \omega} \pm M \left( \frac{1 - v}{1 - \sin \omega} \right)^{\frac{1}{2}} + O(M^2) \quad (B.16)$$

$$v_{3,4}^{(1)} = \frac{1 - \sin \omega}{\cos \omega} \mp M \left( \frac{1 - v}{1 + \sin \omega} \right)^{\frac{1}{2}} + O(M^2) \quad (B.17)$$

The eigenvectors corresponding to  $v_1^{(1)}$  lead to the following relations

$$d\hat{u}_x = d\hat{v}_x = 0, \quad d\tilde{\sigma} = \left( \frac{1 - v}{1 - \sin \omega} \right)^{\frac{1}{2}} d\omega \quad (B.18)$$

In terms of the original variables we find

$$d\omega = \frac{3}{2k} \frac{d\sigma}{L + v} \quad (B.19)$$

$$du_x = \left[ \frac{k}{\mu M} (1 - v)^{\frac{1}{2}} (1 + \sin \omega)^{\frac{1}{2}} - \frac{k v}{\mu} \right] d\omega \quad (B.20)$$

$$dv_x = - \frac{k}{\mu M} (1 - v)^{\frac{1}{2}} (1 - \sin \omega)^{\frac{1}{2}} d\omega \quad (B.21)$$

These expressions can easily be integrated. In the following we list not only the expressions corresponding to  $v_1^{(1)}$  (upper signs), but also the ones corresponding to  $v_4^{(1)}$  (lower signs):

$$u_x = \mp \frac{2k}{\mu M} (1 - v)^{\frac{1}{2}} (1 \mp \sin \omega)^{\frac{1}{2}} \mp \frac{vk}{\mu} \omega + u_x^0 \quad (B.22)$$



$$\sigma = \pm \frac{2}{3} k (1+\nu) \omega + \sigma_0 \quad (\text{B.23})$$

$$v_x = - \frac{2k}{\mu M} (1-\nu)^{\frac{1}{2}} (1 \pm \sin \omega)^{\frac{1}{2}} + v_x^0 \quad (\text{B.24})$$

where  $u_x^0$ ,  $\sigma_0$  and  $v_x^0$  are arbitrary constants.

It is of interest to determine  $\lambda$  from the relation

$$2\lambda k r \sim \frac{1}{\sin \omega} \left( -\sin \theta \partial_\theta v_x + \cos \theta \partial_\theta u_x \right) \quad (\text{B.25})$$

Corresponding to  $v_1^{(1)}$  and  $v_4^{(1)}$  we find

$$\lambda r \sim - \frac{[2(1-\nu)]^{\frac{1}{2}}}{\mu M} < 0 \quad (\text{B.26})$$

Analogously, we find corresponding to  $v_2^{(1)}$  (upper sign) and  $v_3^{(1)}$  (lower sign)

$$\sigma = \pm \frac{2}{3} k (1 + \nu) \omega + \sigma_0 \quad (\text{B.27})$$

$$u_x = \pm \frac{k}{\mu} \left[ \frac{2}{M} (1 \pm \nu)^{\frac{1}{2}} (1 \mp \sin \omega)^{\frac{1}{2}} \mp \nu \omega \right] + u_x^0 \quad (\text{B.28})$$

$$v_x = \frac{2k}{\mu M} (1-\nu)^{\frac{1}{2}} (1 \pm \sin \omega)^{\frac{1}{2}} + v_x^0 \quad (\text{B.29})$$

$$\lambda r \sim \frac{[2(1-\nu)]^{\frac{1}{2}}}{\mu M} > 0 \quad (\text{B.30})$$

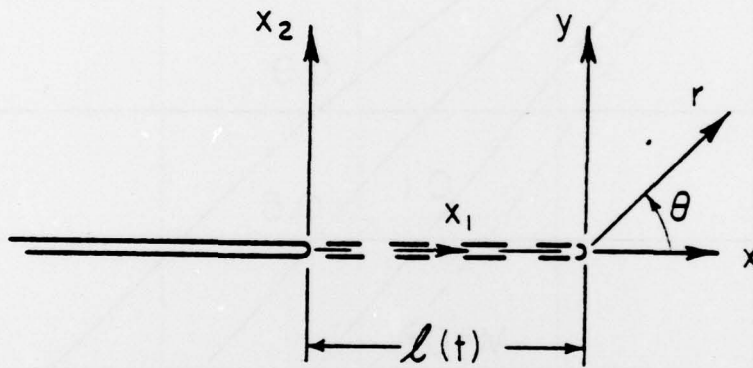


Fig. 1 Propagating crack tip with stationary and moving coordinate systems

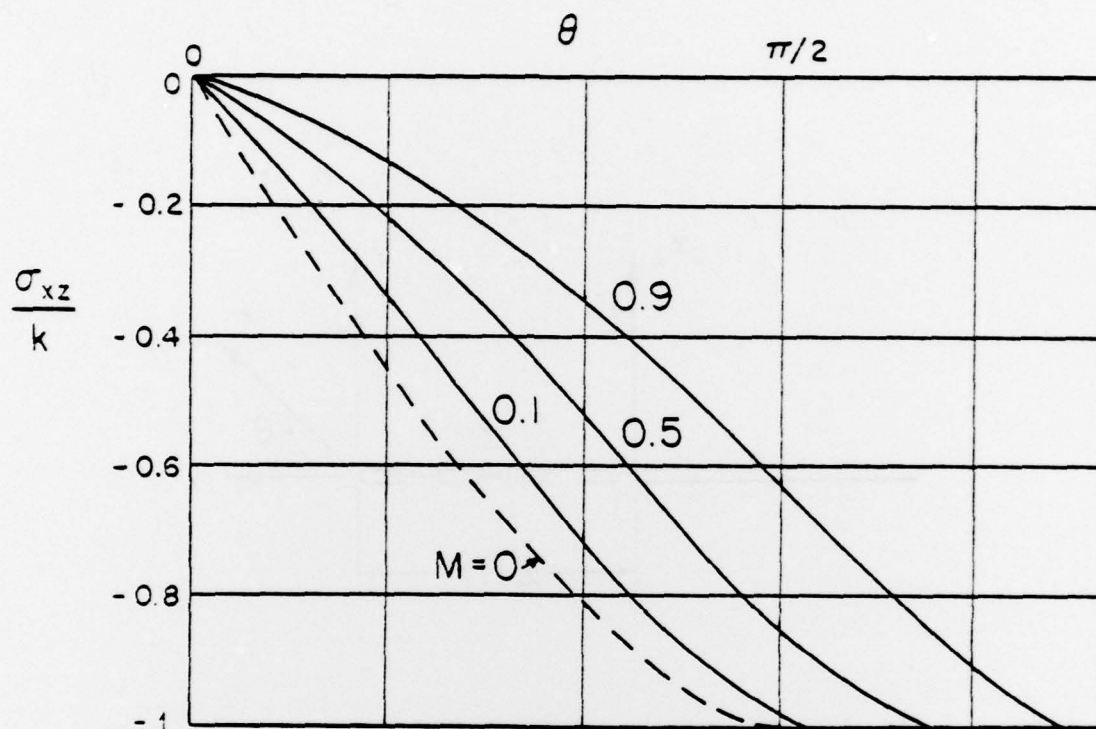


Fig. 2 Dimensionless shear stress,  $\sigma_{xz}/k$ , versus  $\theta$  for various crack-tip speeds;  $k$  = yield stress in pure shear; - - - quasi-static solution.

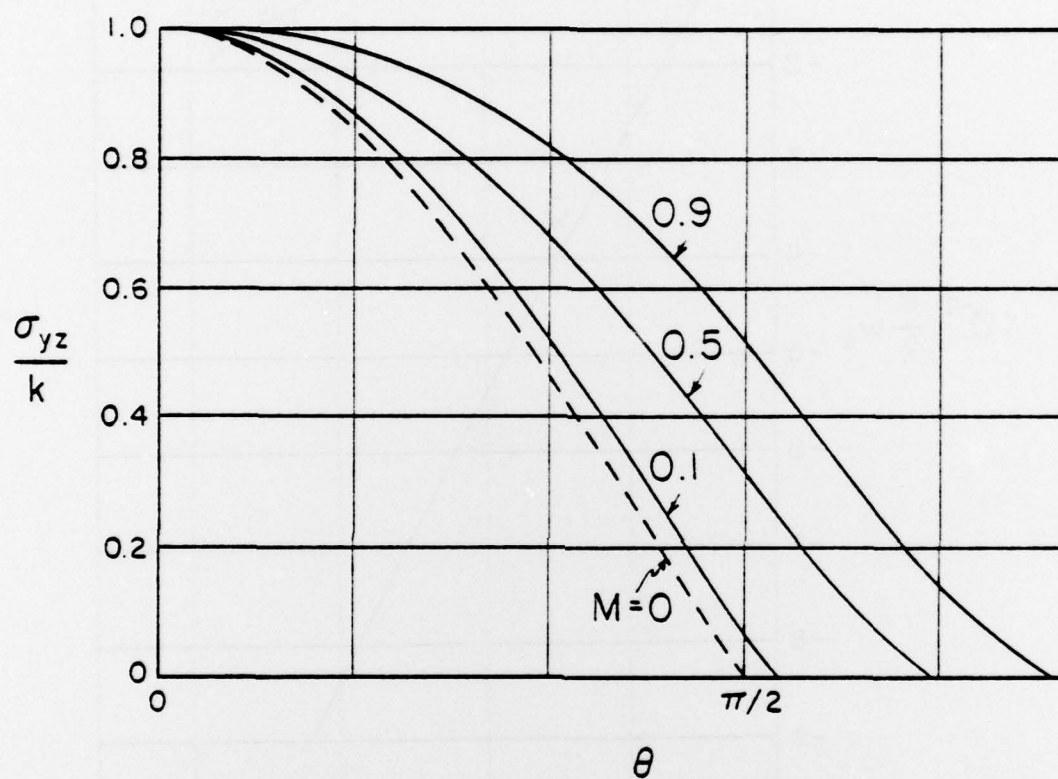


Fig. 3 Dimensionless shear stress,  $\sigma_{yz}/k$ , versus  $\theta$  for various crack-tip speeds;  $k$  = yield stress in pure shear; - - - quasi-static solution.



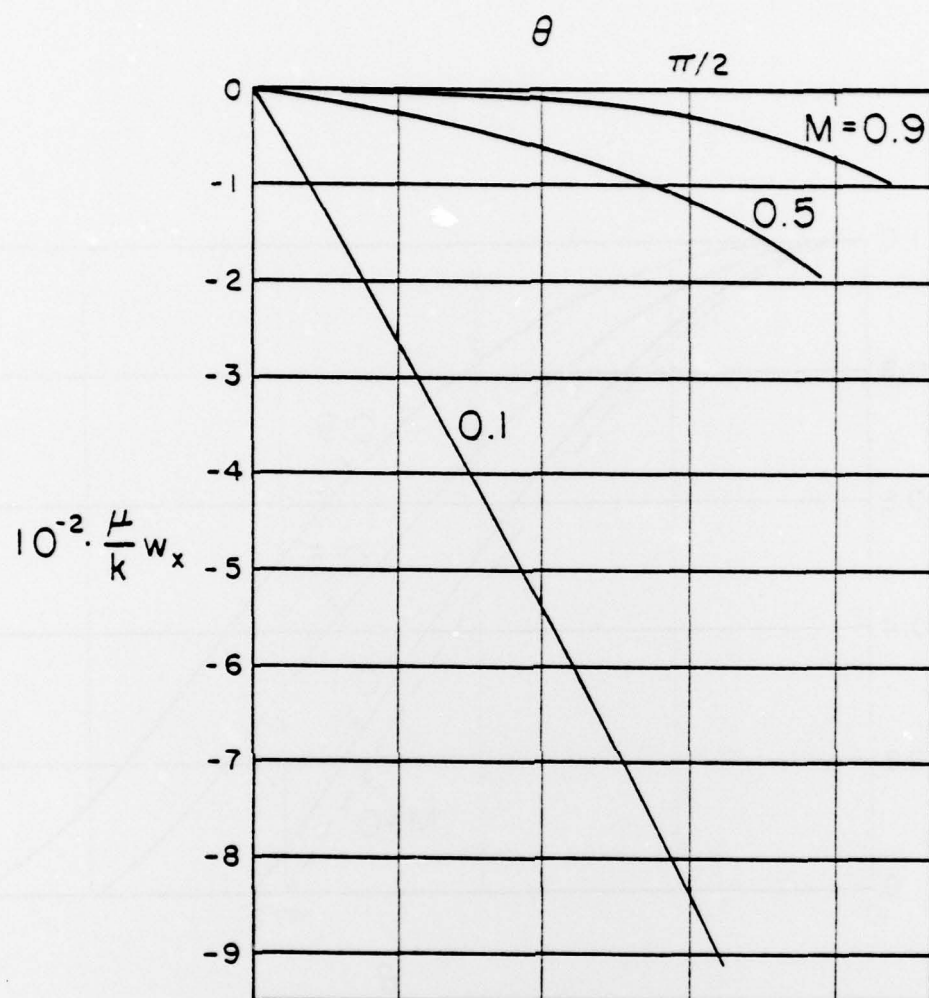


Fig. 4 Engineering shear strain  $(\mu/k)w_x$  versus  $\theta$  in the range  $0 \leq \theta \leq \theta^*$ , for various crack-tip speeds.

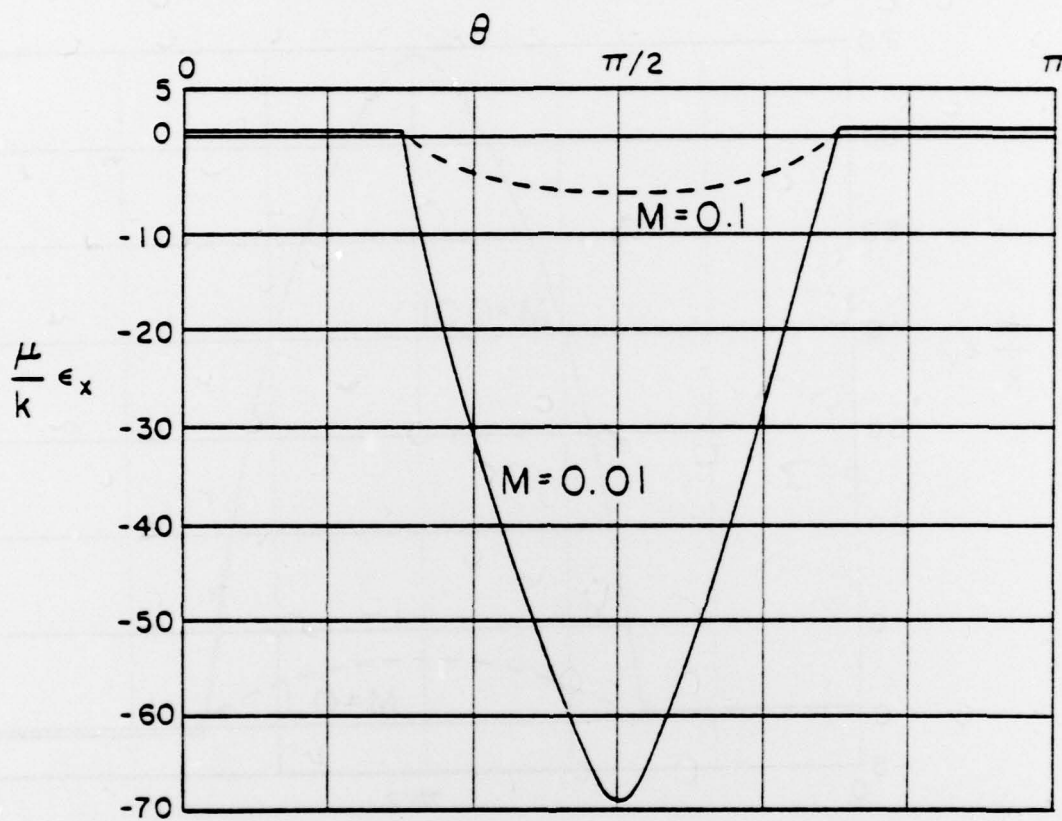


Fig. 5 Strain  $\epsilon_x$  versus  $\theta$  for two crack-tip speeds;  $\nu = 0.3$ .

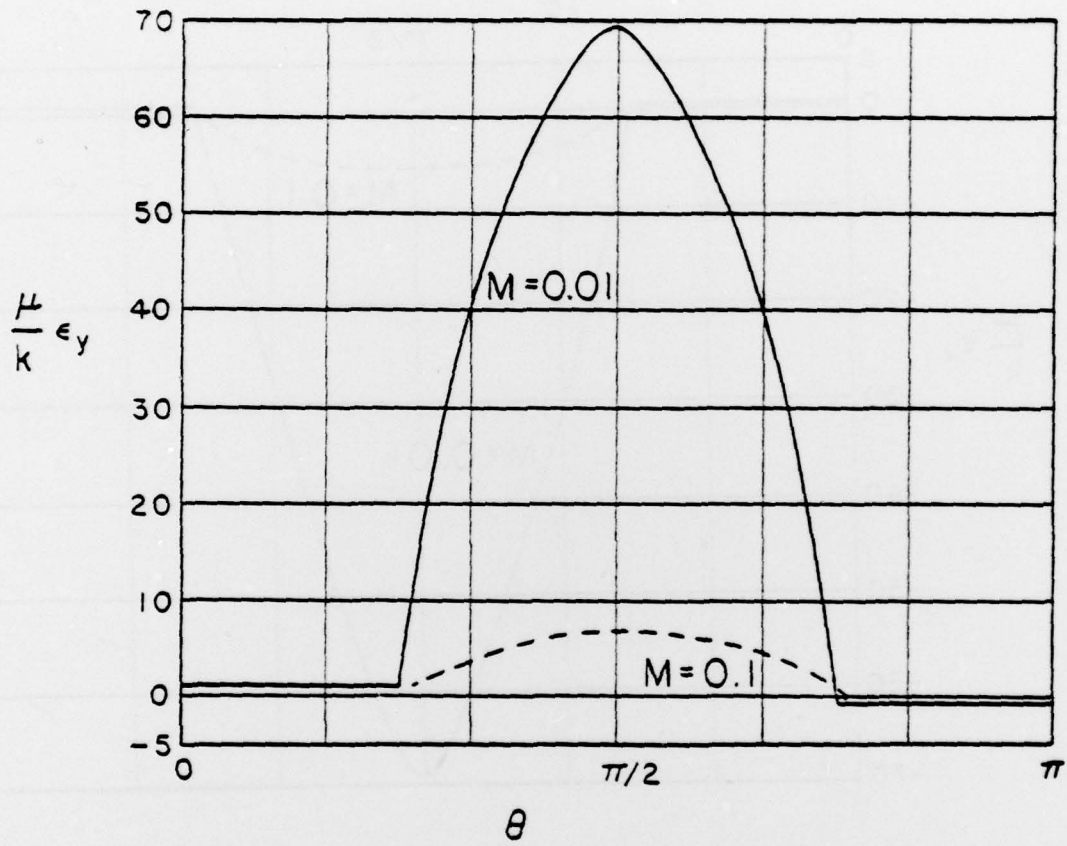


Fig. 6 Strain  $\epsilon_y$  versus  $\theta$  for two crack-tip speeds;  $v = 0.3$ .